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Soliton S matrices for the critical $A_{\mathcal{N}-1}^{(1)}$ chain

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Abstract

We compute by Bethe Ansatz both bulk and boundary hole scattering matrices for the critical $A_{\mathcal{N}-1}^{(1)}$ quantum spin chain. The bulk S matrix coincides with the soliton S matrix for the $A_{\mathcal{N}-1}^{(1)}$ Toda field theory with imaginary coupling. We verify our result for the boundary S matrix using a generalization of the Ghoshal-Zamolodchikov boundary crossing relation.

1 Introduction

Magnetic chains associated with affine Lie algebras [1], [2] constitute large classes of integrable models. These models hold both mathematical and physical interest. Indeed, the models can be shown to have certain quantum-algebra symmetries which are characterized by a parameter q . A variety of powerful analytical techniques can be brought to bear on these models, yielding in the thermodynamic limit exact properties such as excitation spectra, scattering matrices, and correlation functions. These models, which can be regarded as discretized integrable quantum field theories, have numerous applications to statistical mechanics and condensed matter physics.

In a previous paper [3] we began investigating the simplest such class of models, namely, the $A_{\mathcal{N}-1}^{(1)}$ spin chain. This class includes the well-known XXZ chain as the special case $\mathcal{N} = 2$. There we considered the noncritical regime (real q), for which there is a mass gap, and for which the S matrices are characteristic of q -deformed quantum field theory.

In the present work we continue our investigation of the $A_{\mathcal{N}-1}^{(1)}$ spin chain, but consider instead the critical regime ($|q| = 1$). We compute by Bethe Ansatz both bulk and boundary hole scattering matrices. Our result for the bulk S matrix coincides with the soliton S matrix for the $A_{\mathcal{N}-1}^{(1)}$ Toda field theory with imaginary coupling [4]-[6]. We also generalize the Ghoshal-Zamolodchikov [7] boundary crossing relation to the $A_{\mathcal{N}-1}^{(1)}$ case (for which the bulk S matrix does not have crossing symmetry), and use it to help verify our result for the boundary S matrix. For $\mathcal{N} = 2$, the boundary S matrix reduces to the well-known result for the critical XXZ open spin chain [7]-[9]; while in the isotropic limit, it coincides with the result found in [10].

2 The model and its symmetries

We begin by briefly reviewing the construction of the model, both closed (periodic boundary conditions) and open with diagonal boundary fields.¹ For an introduction to the Yang-Baxter equation, integrable spin chains, and the Bethe Ansatz, see e.g. [11], [12].

2.1 Closed chain

The main ingredient for constructing an integrable closed spin chain is a solution R of the Yang-Baxter equation

$$R_{12}(\lambda) R_{13}(\lambda + \lambda') R_{23}(\lambda') = R_{23}(\lambda') R_{13}(\lambda + \lambda') R_{12}(\lambda). \quad (1)$$

We consider here the $A_{\mathcal{N}-1}^{(1)}$ R matrix [12]

$$R(\lambda) = a \sum_j e_{jj} \otimes e_{jj} + b \sum_{j \neq k} e_{jj} \otimes e_{kk} + c \sum_{j \neq k} e_{jk} \otimes e_{kj} e^{-\mu \lambda \operatorname{sign}(j-k)}, \quad (2)$$

¹Our notations closely follow those of Ref. [3], in which the noncritical regime is studied, with the anisotropy parameter η replaced by $i\mu$.

where the indices j, k take values from 1 to \mathcal{N} , and e_{jk} are elementary $\mathcal{N} \times \mathcal{N}$ matrices with matrix elements $(e_{jk})_{ab} = \delta_{ja}\delta_{kb}$. Moreover,

$$a = \sinh(\mu(\lambda + i)), \quad b = \sinh(\mu\lambda), \quad c = \sinh(i\mu), \quad (3)$$

where the anisotropy parameter μ is real. The R matrix becomes $SU(\mathcal{N})$ invariant for $\mu \rightarrow 0$, and is proportional to the permutation matrix for $\lambda = 0$.

The corresponding closed-chain transfer matrix $t(\lambda)$ is given by

$$t(\lambda) = \text{tr}_0 T_0(\lambda), \quad (4)$$

where tr_0 denotes trace over the “auxiliary space” 0, and $T_0(\lambda)$ is the monodromy matrix

$$T_0(\lambda) = R_{0N}(\lambda) \cdots R_{01}(\lambda). \quad (5)$$

We restrict the number of spins, N , to be an even number.

The closed-chain Hamiltonian \mathcal{H} is given by the logarithmic derivative of the transfer matrix at $\lambda = 0$,

$$\mathcal{H} \sim \frac{d}{d\lambda} \log t(\lambda) \Big|_{\lambda=0} = \frac{1}{2} \left(\sum_{n=1}^{N-1} h_{n,n+1} + h_{N,1} \right), \quad (6)$$

where the two-site Hamiltonian h is given by

$$h = - \sum_{j \neq k} e_{jj} \otimes e_{kk} e^{-i\mu \text{sign}(j-k)} + \sum_{j \neq k} e_{jk} \otimes e_{kj}. \quad (7)$$

Here we restrict μ to lie in the range $\mu \in (0, \pi)$.² As for the $A_{\mathcal{N}-1}^{(1)}$ Toda field theory with imaginary coupling, the Hamiltonian for $\mathcal{N} > 2$ is not Hermitian. Nevertheless, we find in our study of low-lying states (see Sec. 3) only real energy eigenvalues.

The transfer matrix has an exact $U(1)^{\mathcal{N}-1}$ symmetry. Indeed, let

$$s^{(k)} = e_{kk} - e_{k+1,k+1}, \quad k = 1, \dots, \mathcal{N} - 1, \quad (8)$$

be the Cartan generators in the defining representation of $SU(\mathcal{N})$; and let $s_n^{(k)}$ be the corresponding generators at site n ,

$$s_n^{(k)} = 1 \otimes \dots \otimes 1 \otimes \overset{n^{th}}{\downarrow} s^{(k)} \otimes 1 \otimes \dots \otimes 1, \quad n = 1, \dots, N. \quad (9)$$

We denote by $S^{(k)}$ the corresponding “total” generators acting on the full space of states

$$S^{(k)} = \sum_{n=1}^N s_n^{(k)}, \quad k = 1, \dots, \mathcal{N} - 1. \quad (10)$$

²The Hamiltonian with $\mu \in (\pi, 2\pi)$ is related to the parity-transformed Hamiltonian with $\mu \in (0, \pi)$. Explicitly, $\mathcal{H}(\mu) = \Pi \mathcal{H}(\mu') \Pi$, where $\mu' = 2\pi - \mu$, and Π is the parity [13] operator. An alternative approach (followed, e.g., in [8],[9]) is to restrict μ even further to the range $\mu \in (0, \frac{\pi}{2})$ and to consider both signs of the Hamiltonian.

The transfer matrix has the symmetry

$$[t(\lambda), S^{(k)}] = 0, \quad k = 1, \dots, \mathcal{N} - 1. \quad (11)$$

The eigenstates and eigenvalues of the transfer matrix have been determined [14],[15],[12] by the nested algebraic Bethe Ansatz method. Indeed, the states are constructed using certain creation operators depending on the solutions $\{\lambda_\alpha^{(j)}\}$ of the Bethe Ansatz equations

$$1 = \prod_{\beta=1}^{M^{(j-1)}} e_{-1}(\lambda_\alpha^{(j)} - \lambda_\beta^{(j-1)}; \mu) \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{M^{(j)}} e_2(\lambda_\alpha^{(j)} - \lambda_\beta^{(j)}; \mu) \prod_{\beta=1}^{M^{(j+1)}} e_{-1}(\lambda_\alpha^{(j)} - \lambda_\beta^{(j+1)}; \mu) \\ \alpha = 1, \dots, M^{(j)}, \quad j = 1, \dots, \mathcal{N} - 1, \quad (12)$$

where

$$e_n(\lambda; \mu) = \frac{\sinh \mu \left(\lambda + \frac{in}{2} \right)}{\sinh \mu \left(\lambda - \frac{in}{2} \right)}, \quad (13)$$

and $M^{(0)} = N$, $M^{(\mathcal{N})} = 0$, $\lambda_\alpha^{(0)} = \lambda_\alpha^{(\mathcal{N})} = 0$. The corresponding energy, momentum, and $S^{(k)}$ eigenvalues are given by

$$E = -\sin^2 \mu \sum_{\alpha=1}^{M^{(1)}} \frac{1}{\cosh(2\mu\lambda_\alpha^{(1)}) - \cos \mu}, \quad (14)$$

$$P = \frac{1}{i} \sum_{\alpha=1}^{M^{(1)}} \log e_1(\lambda_\alpha^{(1)}; \mu) \pmod{2\pi}, \quad (15)$$

$$S^{(k)} = M^{(k-1)} + M^{(k+1)} - 2M^{(k)}. \quad (16)$$

2.2 Open chain

In addition to an R matrix, the construction of an integrable open spin chain requires also a solution K of the boundary Yang-Baxter equation [16]

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \quad (17)$$

We consider here the diagonal $A_{\mathcal{N}-1}^{(1)}$ K matrix [19]

$$K_{(l)}(\lambda, \xi) = \alpha e^{\mu\lambda} \sum_{j=1}^l e_{jj} + \beta e^{-\mu\lambda} \sum_{j=l+1}^{\mathcal{N}} e_{jj}, \quad (18)$$

where

$$\alpha = \sinh(\mu(i\xi - \lambda)), \quad \beta = \sinh(\mu(i\xi + \lambda)), \quad (19)$$

where ξ is a parameter and $l \in \{1, \dots, \mathcal{N} - 1\}$.

The corresponding open-chain transfer matrix $t_{(l)}(\lambda, \xi_-, \xi_+)$ is given by [17]-[19],[3]

$$t_{(l)}(\lambda, \xi_-, \xi_+) = \text{tr}_0 M_0 K_{(l)0}(-\lambda - \rho, \xi_+ - \frac{\mathcal{N}}{2}) T_0(\lambda) K_{(l)0}(\lambda, \xi_-) \hat{T}_0(\lambda), \quad (20)$$

where the monodromy matrix $T_0(\lambda)$ is given by (5), and $\hat{T}_0(\lambda)$ is given by

$$\hat{T}_0(\lambda) = R_{10}(\lambda) \cdots R_{N0}(\lambda). \quad (21)$$

Moreover, M is the matrix in the crossing-unitarity relation

$$R_{12}(-\rho - \lambda)^{t_1} M_1 R_{12}(-\rho + \lambda)^{t_2} M_1^{-1} \propto 1, \quad (22)$$

and is given by the $\mathcal{N} \times \mathcal{N}$ matrix $M_{jk} = \delta_{jk} e^{i\mu(\mathcal{N}-2j+1)}$, and $\rho = i\mathcal{N}/2$. Indeed, it can be shown that this transfer matrix has the commutativity property

$$[t_{(l)}(\lambda, \xi_-, \xi_+), t_{(l)}(\lambda', \xi_-, \xi_+)] = 0, \quad (23)$$

and its derivative at $\lambda = 0$ gives the Hamiltonian,

$$\mathcal{H} \sim \frac{d}{d\lambda} t_{(l)}(\lambda, \xi_-, \xi_+) \Big|_{\lambda=0} = \frac{1}{2} \sum_{n=1}^{N-1} h_{n,n+1} + c(\xi_-) P_{(l)-1} - c(\xi_+ - \mathcal{N} + l) P_{(l)-N}, \quad (24)$$

where the two-site Hamiltonian h is given by (7), and

$$c(\xi) = \frac{1}{4} \sin \mu (i - \cot(\mu \xi)), \quad P_{(l)} = \sum_{j=1}^l e_{jj} - \sum_{j=l+1}^{\mathcal{N}} e_{jj}. \quad (25)$$

In particular, for $\mathcal{N} = 2$ (and therefore $l = 1$), the Hamiltonian is given by [20],[17]

$$\mathcal{H} = \frac{1}{4} \left\{ \sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos \mu \sigma_n^z \sigma_{n+1}^z \right) - \sin \mu \cot(\mu \xi_-) \sigma_1^z + \sin \mu \cot(\mu(\xi_+ - 1)) \sigma_N^z \right\}. \quad (26)$$

Evidently, the parameters ξ_{\pm} correspond to boundary fields.

An important feature [3] of this model is its exact quantum algebra symmetry $U_q(SU(l)) \times U_q(SU(\mathcal{N} - l)) \times U(1)$. Indeed, let $J^{\pm(k)}$ ($k = 1, \dots, \mathcal{N} - 1$) be raising/lowering operators of the quantum algebra $U_q(SU(\mathcal{N}))$ which act on the full space of states, and which obey the commutation relations

$$[J^{+(k)}, J^{-(j)}] = \delta_{k,j} [S^{(k)}]_q, \quad [S^{(k)}, J^{\pm(j)}] = \pm (2\delta_{k,j} - \delta_{k-1,j} - \delta_{k+1,j}) J^{\pm(j)}, \quad (27)$$

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$, and $S^{(k)}$ is given by Eq. (10). For a given value of l ,

$$\begin{aligned} [t_{(l)}(\lambda, \xi_-, \xi_+), S^{(k)}] &= 0, & k = 1, \dots, \mathcal{N} - 1, \\ [t_{(l)}(\lambda, \xi_-, \xi_+), J^{\pm(k)}] &= 0, & k \neq l, \end{aligned} \quad (28)$$

where $q = e^{-i\mu}$. Had we taken the K matrix to be the identity matrix $K(\lambda) = 1$ instead of (18), then the corresponding transfer matrix would have the full $U_q(SU(\mathcal{N}))$ symmetry [18].

The model also has [3] a “duality” symmetry which relates $l \leftrightarrow \mathcal{N} - l$,

$$\mathcal{U}^l t_{(l)}(\lambda, \xi_-, \xi_+) \mathcal{U}^{-l} \propto t_{(\mathcal{N}-l)}(\lambda, -\xi_-, -\xi_+ + \mathcal{N}), \quad (29)$$

where $\mathcal{U}^{\mathcal{N}} = 1$. This is a remnant of the cyclic $(Z_{\mathcal{N}})$ symmetry [12], [21] of the $A_{\mathcal{N}-1}^{(1)}$ R matrix.

The Bethe Ansatz equations are [19]

$$\begin{aligned} 1 &= \left[e_{2\xi_- + l}(\lambda_{\alpha}^{(l)}; \mu) e_{-(2\xi_+ - 2\mathcal{N} + l)}(\lambda_{\alpha}^{(l)}; \mu) \delta_{l,j} + (1 - \delta_{l,j}) \right] \\ &\times \prod_{\beta=1}^{M^{(j-1)}} e_{-1}(\lambda_{\alpha}^{(j)} - \lambda_{\beta}^{(j-1)}; \mu) e_{-1}(\lambda_{\alpha}^{(j)} + \lambda_{\beta}^{(j-1)}; \mu) \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{M^{(j)}} e_2(\lambda_{\alpha}^{(j)} - \lambda_{\beta}^{(j)}; \mu) e_2(\lambda_{\alpha}^{(j)} + \lambda_{\beta}^{(j)}; \mu) \\ &\times \prod_{\beta=1}^{M^{(j+1)}} e_{-1}(\lambda_{\alpha}^{(j)} - \lambda_{\beta}^{(j+1)}; \mu) e_{-1}(\lambda_{\alpha}^{(j)} + \lambda_{\beta}^{(j+1)}; \mu) \\ &\alpha = 1, \dots, M^{(j)}, \quad j = 1, \dots, \mathcal{N} - 1. \end{aligned} \quad (30)$$

The energy is given by (14) (plus terms that are independent of $\{\lambda_{\alpha}^{(j)}\}$), and the $S^{(k)}$ eigenvalues are again given by Eq. (16).

3 Bulk S matrix

In order to investigate bulk properties, we consider the periodic chain with Bethe Ansatz Eqs. (12). We assume that the ground state is the Bethe Ansatz state with no holes, i.e., with $\mathcal{N} - 1$ filled real Fermi seas. Holes in these seas correspond to “solitons”. Indeed, the Bethe Ansatz state with one hole in the j^{th} sea is a particle-like excited state which belongs to the fundamental representation $[j]$ of $U_q(SU(\mathcal{N}))$, corresponding to a Young tableau with a single column of j boxes. Such an excitation with rapidity $\tilde{\lambda}^{(j)}$ has energy $\frac{\pi \sin \mu}{\mu} s^{(j)}(\tilde{\lambda}^{(j)})$ and momentum $p^{(j)}(\tilde{\lambda}^{(j)})$, where

$$s^{(j)}(\lambda) = \frac{1}{\mathcal{N}} \frac{\sin(\frac{\pi}{\mathcal{N}}(\mathcal{N} - j))}{\cos(\frac{\pi}{\mathcal{N}}(\mathcal{N} - j)) + \cosh(\frac{2\pi\lambda}{\mathcal{N}})}, \quad (31)$$

and the momentum satisfies

$$\frac{1}{2\pi} \frac{d}{d\lambda} p^{(j)}(\lambda) = s^{(j)}(\lambda). \quad (32)$$

Note that $s^{(j)}(\lambda)$ has the periodicity $\lambda \rightarrow \lambda + i\mathcal{N}$, which suggests that the physical strip is $0 \leq \Im m \lambda \leq \frac{\mathcal{N}}{2}$.

Exact bulk scattering matrices can be computed using a generalization of the method of Korepin [22] and Andrei-Destri [23]. We define the two-particle S matrix $R^{[j] \otimes [k]}$ for particles of type $[j]$ and $[k]$ by the momentum quantization condition

$$\left(e^{ip^{(j)}(\tilde{\lambda}^{(j)})N} R^{[j] \otimes [k]} - 1 \right) |\tilde{\lambda}^{(j)}, \tilde{\lambda}^{(k)}\rangle = 0. \quad (33)$$

For the scalar factor, we obtain (cf. [10])

$$\begin{aligned} R_0^{[j] \otimes [k]} &\sim \exp \left\{ i2\pi N \int_{-\infty}^{\tilde{\lambda}^{(j)}} \left(\sigma^{(j)}(\lambda) - s^{(j)}(\lambda) \right) d\lambda \right\} \\ &= \exp \left\{ 2 \int_0^\infty \frac{d\omega}{\omega} \sinh \left(i\omega(\tilde{\lambda}^{(j)} - \tilde{\lambda}^{(k)}) \right) \left(\delta_{jk} - \hat{R}_{jk}(\omega) \right) \right\}, \end{aligned} \quad (34)$$

where $\sigma^{(j)}(\lambda)$ is the density of Bethe Ansatz roots $\{\lambda_\alpha^{(j)}\}$ for the state with holes of rapidities $\tilde{\lambda}^{(j)}$ and $\tilde{\lambda}^{(k)}$ in the j^{th} and k^{th} seas, respectively; and (see, e.g., [12])

$$\hat{R}_{jj'}(\omega) = \frac{\sinh\left(\frac{\nu\omega}{2}\right) \sinh\left(\frac{j_<\omega}{2}\right) \sinh\left((\mathcal{N} - j_>)\frac{\omega}{2}\right)}{\sinh\left((\nu - 1)\frac{\omega}{2}\right) \sinh\left(\frac{\mathcal{N}\omega}{2}\right) \sinh\left(\frac{\omega}{2}\right)}, \quad (35)$$

where $\nu = \frac{\pi}{\mu}$, $j_> = \max(j, j')$ and $j_< = \min(j, j')$. Note that $\mu \in (0, \pi)$ implies $\nu > 1$.

Let us focus on the particular case of two holes of type [1]. The scalar factor is given by

$$R_0^{[1] \otimes [1]} = \exp \left\{ 2 \int_0^\infty \frac{d\omega}{\omega} \sinh(i\omega\tilde{\lambda}) \frac{\sinh\left(\frac{\omega}{2}\right) \sinh\left((\nu - \mathcal{N})\frac{\omega}{2}\right)}{\sinh\left((\nu - 1)\frac{\omega}{2}\right) \sinh\left(\frac{\mathcal{N}\omega}{2}\right)} \right\}, \quad (36)$$

where $\tilde{\lambda} = \tilde{\lambda}_1^{(1)} - \tilde{\lambda}_2^{(1)}$. Moreover, let us consider the following Bethe Ansatz states:

- (a) two holes and one 2-string (4-string) of positive parity in the first sea if $\nu > \frac{3}{2}$ ($\nu < \frac{3}{2}$, respectively)
- (b) two holes and one 1-string (3-string) of negative parity in the first sea if $\nu > 2$ ($\nu < 2$, respectively)

The corresponding S matrix elements are given by ³

$$R_{(a)} = \frac{\sin\left(\frac{\pi}{2(\nu-1)}(1 - i\tilde{\lambda})\right)}{\sin\left(\frac{\pi}{2(\nu-1)}(1 + i\tilde{\lambda})\right)} R_0^{[1] \otimes [1]}, \quad R_{(b)} = -\frac{\cos\left(\frac{\pi}{2(\nu-1)}(1 - i\tilde{\lambda})\right)}{\cos\left(\frac{\pi}{2(\nu-1)}(1 + i\tilde{\lambda})\right)} R_0^{[1] \otimes [1]}. \quad (37)$$

It still remains an open problem to determine all \mathcal{N}^2 two-particle states. However, let us assume that the scattering matrix $R^{[1] \otimes [1]}$ has the same structure as the R matrix (2); i.e.,

$$R(\tilde{\lambda})^{[1] \otimes [1]} = \mathbf{a} \sum_j e_{jj} \otimes e_{jj} + \mathbf{b} \sum_{j \neq k} e_{jj} \otimes e_{kk} + \mathbf{c} \sum_{j \neq k} e_{jk} \otimes e_{kj} e^{-x\tilde{\lambda} \text{ sign}(j-k)}, \quad (38)$$

³Up to signs, which are not easy to obtain from the Bethe Ansatz, but which can be fixed by requiring that for $\tilde{\lambda} = 0$ the amplitudes be equal to +1 and -1 respectively; i.e., that the matrix $R(0)$ (see below) be equal to the permutation matrix.

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, x$ are to be determined. This matrix has the three distinct eigenvalues $\mathbf{a}, \mathbf{b} + \mathbf{c}, \mathbf{b} - \mathbf{c}$, which we identify with the three amplitudes $\mathbf{R}_0^{[1] \otimes [1]}, \mathbf{R}_{(a)}, \mathbf{R}_{(b)}$, respectively. Note that the eigenvalues of $\mathbf{R}^{[1] \otimes [1]}$ are independent of x . That is, the value of x cannot be determined by considering Bethe Ansatz states with two holes. We fix the value of x (up to a sign) by requiring that $\mathbf{R}^{[1] \otimes [1]}$ obey the Yang-Baxter Eq. (1).⁴ In this way, we obtain

$$\mathbf{a} = \sin\left(\frac{\pi}{\nu-1}(1+i\tilde{\lambda})\right)r(\tilde{\lambda}), \quad \mathbf{b} = -\sin\left(\frac{i\pi\tilde{\lambda}}{\nu-1}\right)r(\tilde{\lambda}), \quad \mathbf{c} = \sin\left(\frac{\pi}{\nu-1}\right)r(\tilde{\lambda}), \quad x = \frac{\pi}{\nu-1}, \quad (39)$$

where

$$r(\tilde{\lambda}) = \frac{\mathbf{R}_0^{[1] \otimes [1]}}{\sin\left(\frac{\pi}{\nu-1}(1+i\tilde{\lambda})\right)}. \quad (40)$$

For $\mathcal{N} = 2$, this S matrix agrees with the known result for the XXZ chain [25], [13]. Moreover, the scalar factor (36) coincides with the expression given in Eq. (3.29) of Ref. [6] for the $a_n^{(1)}$ scalar factor in affine Toda field theory with imaginary coupling, provided we make the following identifications:

$$n \leftrightarrow \mathcal{N} - 1, \quad \lambda \leftrightarrow \frac{1}{\nu-1}, \quad \mu \leftrightarrow \frac{-i\tilde{\lambda}}{\nu-1}. \quad (41)$$

It has been observed [4]-[6] that (after a certain gauge transformation) this S matrix has a $U_q(SU(\mathcal{N}))$ symmetry with $q = e^{\frac{i\pi}{\nu-1}}$. On the other hand, from (28) we see that the finite-size transfer matrix has an “approximate” $U_q(SU(\mathcal{N}))$ symmetry with $q = e^{\frac{i\pi}{\nu}}$. Evidently, as a result of filling the Fermi seas and taking the thermodynamic limit, the value of q becomes renormalized. (See also e.g. [26].)

4 Boundary S matrix

We define the boundary S matrices $\mathbf{K}_{(l) [j]}^{\mp}$ for a particle of type $[j]$ by the quantization condition [27]

$$\left(e^{i2p^{(j)}(\tilde{\lambda}^{(j)})N} \mathbf{K}_{(l) [j]}^{+} \mathbf{K}_{(l) [j]}^{-} - 1\right) |\tilde{\lambda}^{(j)}\rangle = 0. \quad (42)$$

For simplicity, we consider only the case of a hole of type $[1]$. The quantum-algebra symmetry (28) implies that the boundary S matrices $\mathbf{K}_{(l) [1]}^{\mp}$ are diagonal $\mathcal{N} \times \mathcal{N}$ matrices of the same form as the K matrix (18),

$$\mathbf{K}_{(l) [1]}^{\mp} = \alpha_{(l)}^{\mp} e^{\pm y \tilde{\lambda}^{(1)}} \sum_{j=1}^l e_{jj} + \beta_{(l)}^{\mp} e^{\mp y \tilde{\lambda}^{(1)}} \sum_{j=l+1}^{\mathcal{N}} e_{jj}, \quad (43)$$

⁴Presumably, one can alternatively consider Bethe Ansatz states with three holes, and require factorization of the three-particle S matrix into a product of two-particle S matrices, as was done for the XXX model in [24].

where $\alpha_{(l)}^\mp, \beta_{(l)}^\mp, y$ are to be determined.⁵ We have the relation

$$\alpha_{(l)}^+ \alpha_{(l)}^- \sim \exp \left\{ i 2\pi N \int_0^{\tilde{\lambda}^{(1)}} \left(\sigma_{(l)}^{(1)}(\lambda) - 2s^{(1)}(\lambda) \right) d\lambda \right\}, \quad (44)$$

where $\sigma_{(l)}^{(1)}(\lambda)$ is the density of Bethe Ansatz roots $\{\lambda_\alpha^{(1)}\}$ for the state with one hole of rapidity $\tilde{\lambda}^{(1)}$ in the first sea. From the Bethe Ansatz Eqs. (30), we find (cf. [3],[10])

$$\alpha_{(l)}^- = i \cosh \left(\frac{\pi}{\nu-1} (\tilde{\lambda}^{(1)} - \frac{i}{2}(\nu-2\xi_-)) \right) k(\tilde{\lambda}^{(1)}, \xi_-), \quad (45)$$

where

$$k(\tilde{\lambda}, \xi) = k_0(\tilde{\lambda}) k_1(\tilde{\lambda}, \xi), \quad (46)$$

with

$$\begin{aligned} k_0(\tilde{\lambda}) &= \exp \left\{ 2 \int_0^\infty \frac{d\omega}{\omega} \sinh(2i\omega\tilde{\lambda}) \frac{\sinh((\mathcal{N}+1)\frac{\omega}{2}) \sinh((\nu-\mathcal{N})\frac{\omega}{2})}{\sinh(\mathcal{N}\omega) \sinh((\nu-1)\frac{\omega}{2})} \right\}, \\ k_1(\tilde{\lambda}, \xi) &= \frac{1}{i \cosh \left(\frac{\pi}{\nu-1} (\tilde{\lambda} - \frac{i}{2}(\nu-2\xi)) \right)} \\ &\times \exp \left\{ -2 \int_0^\infty \frac{d\omega}{\omega} \sinh(2i\omega\tilde{\lambda}) \frac{\sinh((\mathcal{N}-l)\omega) \sinh((\nu-2\xi-l)\omega)}{\sinh((\nu-1)\omega) \sinh(\mathcal{N}\omega)} \right\}. \end{aligned} \quad (47)$$

Moreover, with the help of the “duality” symmetry (29), we obtain

$$\beta_{(l)}^- = i \cosh \left(\frac{\pi}{\nu-1} (\tilde{\lambda}^{(1)} + \frac{i}{2}(\nu-2\xi_-)) \right) k(\tilde{\lambda}^{(1)}, \xi_-). \quad (48)$$

Our approach for computing $\alpha_{(l)}^-$ and $\beta_{(l)}^-$ involves performing certain Fourier transformations, which leads to the following restriction of parameters: $\mathcal{N} < 2\xi_- + l < 2\nu$. In particular, this requires $\nu > \frac{\mathcal{N}}{2}$.

The parameter y can be determined using the boundary Yang-Baxter Eq. (17) (in parallel with our analysis of the parameter x in the bulk S matrix), and we obtain the value

$$y = \frac{\pi}{\nu-1}. \quad (49)$$

Our expression for the boundary S matrix agrees with [7]-[9]⁶ for $\mathcal{N} = 2$, and with [10] in the isotropic limit $\nu \rightarrow \infty$. A further check on our result is performed in the following section. Finally, we remark that for $\mathbf{K}_{(l)}^+$ [1] there are similar expressions involving ξ_+ .

⁵The factors $e^{\pm y \tilde{\lambda}^{(1)}}$ in the boundary S matrices were unfortunately omitted in [3].

⁶In [9] we use a slightly different definition of the boundary parameters ξ_\mp .

5 Functional equations for boundary S matrix

Thus far, we have pursued the direct Bethe-Ansatz calculation of S matrices. Alternatively, S matrices can be found (up to CDD ambiguities) by the “bootstrap” approach [28]. Indeed, the bulk S matrix scalar factor (36) was first obtained for affine Toda theory in this way [4]-[6].

A natural question is whether the boundary S matrix can also be obtained by a bootstrap approach. For the case that the bulk S matrix has crossing symmetry, Ghoshal and Zamolodchikov [7] have formulated a boundary crossing relation, which – together with the boundary unitarity relation – determines the scalar factor of the boundary S matrix, up to a boundary CDD ambiguity. Unfortunately, the $A_{\mathcal{N}-1}^{(1)}$ bulk S matrix does not have crossing symmetry for $\mathcal{N} > 2$, and hence, the Ghoshal-Zamolodchikov relation cannot be directly applied.

In this section, we generalize the Ghoshal-Zamolodchikov boundary crossing relation to the $A_{\mathcal{N}-1}^{(1)}$ case, and we find that the boundary S matrix scalar factor (46),(47) can indeed be obtained by this bootstrap approach. The basic observation is that, although the bulk S matrix lacks crossing symmetry, it does have crossing-unitarity symmetry; and one expects that a corresponding property should hold for the boundary S matrix.⁷

We begin by briefly reviewing the case that the bulk S matrix does have crossing symmetry, and rewriting the Ghoshal-Zamolodchikov boundary crossing relation in matrix form. We assume that the bulk S matrix $R(\tilde{\lambda})$ has PT symmetry,

$$R_{12}(\tilde{\lambda})^{t_1 t_2} = R_{21}(\tilde{\lambda}), \quad (50)$$

where $R_{21} = \mathcal{P}_{12} R_{12} \mathcal{P}_{12}$, \mathcal{P} is the permutation matrix, and t_i denotes transposition in the i^{th} space. We write the bulk crossing relation as

$$R_{12}(\tilde{\lambda})^{t_2} = V_1 R_{12}(\rho - \tilde{\lambda}) V_1, \quad (51)$$

where the crossing matrix V satisfies $V^2 = 1$, and ρ is the crossing parameter. The unitarity relation is

$$R_{12}(\tilde{\lambda}) R_{21}(-\tilde{\lambda}) = 1. \quad (52)$$

Combining the three relations (50)-(52), one obtains the crossing-unitarity relation

$$R_{12}(\rho - \tilde{\lambda})^{t_1} M_1 R_{12}(\rho + \tilde{\lambda})^{t_2} M_1^{-1} = 1, \quad (53)$$

where $M = V^t V$. We observe that the Ghoshal-Zamolodchikov boundary crossing relation [7] for the boundary S matrix $K(\tilde{\lambda})$ is consistent with

$$\text{tr}_1 \left[\check{R}_{21}(-2\tilde{\lambda}) M_1 K_1(\rho + \tilde{\lambda}) \right] = \left(V_2 K_2(-\tilde{\lambda}) V_2 \right)^{t_2}, \quad (54)$$

⁷A different generalization of the Ghoshal-Zamolodchikov boundary crossing relation has recently been discussed in Ref. [29].

where $\check{R}_{12} = \mathcal{P}_{12}R_{12}$, and hence $\check{R}_{21} = R_{12}\mathcal{P}_{12}$. (For aesthetic reasons, we have made a shift in the rapidity variable by $\rho/2$, but this is not necessary.)

With the help of the boundary unitarity relation

$$K(\tilde{\lambda})K(-\tilde{\lambda}) = 1, \quad (55)$$

one can now obtain the relation

$$\text{tr}_{13} \left[\check{R}_{21}(-2\tilde{\lambda})M_1K_1(\rho + \tilde{\lambda})\check{R}_{23}(2\tilde{\lambda})M_3K_3(\rho - \tilde{\lambda}) \right] = 1. \quad (56)$$

We shall see below that this relation, which is a boundary generalization of the bulk crossing-unitarity relation (53), is the desired result.

Let us now turn to the $A_{\mathcal{N}-1}^{(1)}$ bulk S matrix $R^{[1]\otimes[1]}$. Although it does not have crossing symmetry, it does obey the crossing-unitarity relation (53) with $M_{jk} = \delta_{jk}e^{-i(\mathcal{N}-2j+1)\frac{\pi}{\nu-1}}$, and $\rho = i\mathcal{N}/2$ (cf. Eq. (22)). We observe that this relation together with the unitarity relation (52) determine the scalar factor of the bulk S matrix. Indeed, substituting the form (38), (39) into these relations, we obtain the following functional equations for $r(\tilde{\lambda})$:

$$\begin{aligned} r\left(\frac{i\mathcal{N}}{2} + \tilde{\lambda}\right) r\left(\frac{i\mathcal{N}}{2} - \tilde{\lambda}\right) &= \frac{1}{\sin\left(\frac{\pi}{\nu-1}\left(\frac{\mathcal{N}}{2} + i\tilde{\lambda}\right)\right) \sin\left(\frac{\pi}{\nu-1}\left(\frac{\mathcal{N}}{2} - i\tilde{\lambda}\right)\right)}, \\ r(\tilde{\lambda}) r(-\tilde{\lambda}) &= \frac{1}{\sin\left(\frac{\pi}{\nu-1}(1 + i\tilde{\lambda})\right) \sin\left(\frac{\pi}{\nu-1}(1 - i\tilde{\lambda})\right)}. \end{aligned} \quad (57)$$

Solving these equations yields the result (40), (36), up to a CDD factor.

Finally, we turn to the $A_{\mathcal{N}-1}^{(1)}$ boundary S matrix $K_{(l)}^-$ [1]. Substituting the form (43), (45), (48) into the boundary crossing-unitarity relation (56), we obtain a functional equation for the scalar factor $k(\tilde{\lambda}, \xi)$:

$$\begin{aligned} k\left(\frac{i\mathcal{N}}{2} + \tilde{\lambda}, \xi\right) k\left(\frac{i\mathcal{N}}{2} - \tilde{\lambda}, \xi\right) &= \frac{\sin\left(\frac{\pi}{\nu-1}(1 + 2i\tilde{\lambda})\right) \sin\left(\frac{\pi}{\nu-1}(1 - 2i\tilde{\lambda})\right)}{\sin\left(\frac{\pi}{\nu-1}(\mathcal{N} + 2i\tilde{\lambda})\right) \sin\left(\frac{\pi}{\nu-1}(\mathcal{N} - 2i\tilde{\lambda})\right)} \\ &\times \frac{(-1)}{\cosh\left(\frac{\pi}{\nu-1}\left(\tilde{\lambda} - \frac{i}{2}(\nu - 2\xi + \mathcal{N} - 2l)\right)\right) \cosh\left(\frac{\pi}{\nu-1}\left(\tilde{\lambda} + \frac{i}{2}(\nu - 2\xi + \mathcal{N} - 2l)\right)\right)} \end{aligned} \quad (58)$$

(up to a rapidity-independent phase, which can be absorbed by a redefinition of M). Similarly, the boundary unitarity relation (55) implies

$$k(\tilde{\lambda}, \xi) k(-\tilde{\lambda}, \xi) = -\frac{1}{\cosh\left(\frac{\pi}{\nu-1}\left(\tilde{\lambda} - \frac{i}{2}(\nu - 2\xi)\right)\right) \cosh\left(\frac{\pi}{\nu-1}\left(\tilde{\lambda} + \frac{i}{2}(\nu - 2\xi)\right)\right)}. \quad (59)$$

Solving these equations yields the result (46), (47) up to a CDD factor.

6 Discussion

The critical $A_{N-1}^{(1)}$ quantum spin chain shares a number of features with $A_{N-1}^{(1)}$ Toda quantum field theory with imaginary coupling. In particular, the two models have solitonic excitations with the same bulk S matrix. This suggests that the spin chain may be a type of discretization (see, e.g., [26],[30]) of the affine Toda theory. This further suggests that it may be possible to find “soliton preserving” integrable boundary conditions for the affine Toda theory which lead to the boundary S matrix which we have given here. Searches for such integrable boundary conditions have not yet been successful (see, e.g., [31],[32]), but progress in this direction has recently been reported [33]. (Boundary S matrices with a structure similar to (43) have recently been proposed for the $O(N)$ Gross-Neveu and nonlinear sigma models [34] and for the $SO(N)$ principal chiral model [35].) As in the affine Toda theory, the spin chain has various bound states, on which we hope to report soon.

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References

- [1] V.V. Bazhanov, Phys. Lett. *159B* (1985) 321; Commun. Math. Phys. *113* (1987) 471.
- [2] M. Jimbo, Commun. Math. Phys. *102* (1986) 537; *Lecture Notes in Physics*, Vol. 246, (Springer, 1986), p. 335.
- [3] A. Doikou and R.I. Nepomechie, Nucl. Phys. *B530* (1998) 641.
- [4] D. Bernard and A. LeClair, Commun. Math. Phys. *142* (1991) 99.
- [5] T. Hollowood, Int. J. Mod. Phys. *A8* (1993) 947.
- [6] G.M. Gandenberger, Int. J. Mod. Phys. *A13* (1998) 4553.
- [7] S. Ghoshal and A. B. Zamolodchikov, Int. J. Mod. Phys. *A9* (1994) 3841.
- [8] P. Fendley and H. Saleur, Nucl. Phys. *B428* (1994) 681.
- [9] A. Doikou and R.I. Nepomechie, J. Phys. *A32* (1999) 3663.
- [10] A. Doikou and R.I. Nepomechie, Nucl. Phys. *B521* (1998) 547.
- [11] L.D. Faddeev and L.A. Takhtajan, Russ. Math. Surv. *34* (1979) 11; P.P. Kulish and E.K. Sklyanin, in *Lecture Notes in Physics*, v. 151, (Springer, 1982) 61; V.E. Korepin, N.M. Bogoliubov, and A.G. Izergin, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge University Press, 1993).

- [12] H.J. de Vega, *Int. J. Mod. Phys. A* **4** (1989) 2371.
- [13] A. Doikou and R.I. Nepomechie, *J. Phys. A* **31** (1998) L621.
- [14] B. Sutherland, *Phys. Rev. B* **12** (1975) 3795.
- [15] P.P. Kulish and N.Yu. Reshetikhin, *Sov. Phys. JETP* **53** (1981) 108.
- [16] I.V. Cherednik, *Theor. Math. Phys.* **61** (1984) 977.
- [17] E.K. Sklyanin, *J. Phys. A* **21** (1988) 2375.
- [18] L. Mezincescu and R.I. Nepomechie, *J. Phys. A* **24** (1991) L17; *Int. J. Mod. Phys. A* **7** (1992) 5657.
- [19] H.J. de Vega and A. González-Ruiz, *Nucl. Phys. B* **417** (1994) 553; *Mod. Phys. Lett. A* **9** (1994) 2207.
- [20] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, *J. Phys. A* **20** (1987) 6397.
- [21] A.A. Belavin, *Funct. Anal. Appl.* **14** (1980) 260.
- [22] V.E. Korepin, *Theor. Math. Phys.* **41** (1979) 953.
- [23] N. Andrei and C. Destri, *Nucl. Phys. B* **231** (1984) 445.
- [24] A. Doikou, L. Mezincescu and R.I. Nepomechie, *Mod. Phys. Lett. A* **12** (1997) 2591.
- [25] A.N. Kirillov and N.Yu. Reshetikhin, *J. Phys. A* **20** (1987) 1565.
- [26] C. Destri and H.J. de Vega, *Nucl. Phys. B* **374** (1992) 692; *B* **385** (1992) 361.
- [27] M. Grisaru, L. Mezincescu and R.I. Nepomechie, *J. Phys. A* **28** (1995) 1027; A. Doikou, L. Mezincescu and R.I. Nepomechie, *J. Phys. A* **30** (1997) L507.
- [28] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Ann. Phys.* **120** (1979) 253; A.B. Zamolodchikov, *Sov. Sci. Rev. A* **2** (1980) 1.
- [29] B-Y Hou, K-J Shi and W-L Yang, *Commun. Theor. Phys.* **30** (1998) 415.
- [30] C. Destri and H.J. de Vega, *Nucl. Phys. B* **290** (1987) 363; *J. Phys. A* **22** (1989) 1329.
- [31] E. Corrigan, hep-th/9612138
- [32] G.M. Gandenberger, *Nucl. Phys. B* **542** (1999) 659.
- [33] G.W. Delius, *Phys. Lett. B* **444** (1998) 217.
- [34] A. De Martino and M. Moriconi, *Phys. Lett. B* **451** (1999) 354.
- [35] N.J. MacKay, *J. Phys. A* **32** (1999) L189.